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CAYLEY'S PAPERS.

The Collected Mathematical Papers of Arthur Cayley, Sc.D., F.R.S. Vols. viii., ix. Pp. liv + 570, xvi + 622. (Cambridge: at the University Press, 1895, 1896.)

THESE two volumes form the first of those published after Cayley's death in 1895. The first thirty-eight sheets of Vol. viii. were revised by the author, who added a note on one paper (No. 518); the duty of editing the rest of the papers was entrusted to Prof. Forsyth, who has very faithfully carried out the plan and arrangements which, in the absence of definite instructions, he was able to infer from the previous volumes.

Perhaps the reader's first impression after surveying these 144 papers, mostly published in the years 1871–77, is that they are very miscellaneous, and that comparatively few are of paramount importance. The fact is that Cayley is, as it were, brought into unfavourable comparison with himself; short notes on special problems of geometry and analysis, and solutions of Smith's Prize papers cannot rank with the immortal "Memoirs on Quantics," or some of the earlier geometrical papers, such as that upon plane cubic curves. But it is unreasonable to expect an artist to produce an uninterrupted succession of masterpieces; and it is to be remembered that Cayley seldom, if ever, wrote upon any subject without developing some instructive point or giving an example of his own characteristic elegance.

In trying to give some account of the more important of these memoirs it will be convenient to take the geometry and the analysis separately. Not that the boundary line is very easy to fix: Cayley was never a geometrician in the sense in which the word may be applied to Apollonius or Steiner. But some of the papers have an interest mainly geometrical, although the methods used are almost wholly algebraic; and with them we will begin.

Perhaps the most important are those which deal with transformation, correspondence, and the singularities of algebraical curves and surfaces. With these difficult theories Cayley dealt in a masterly way: he avoided, as if by instinct, the many opportunities of mistake which present themselves in a method which is largely enumerative, and he had the gift of predicting general results from the consideration of special cases.

Coming next to what may be called the metrical geometry of surfaces, which has developed so greatly in recent years, we have papers on curves of curvature, on geodesics on quadrics, and on orthogonal surfaces. To this group may perhaps be added a paper on evolutes and parallel curves, though this is rather meant to illustrate the non-Euclidian geometry.

There are three monographs, on Steiner's surface, on the centro-surface of an ellipsoid, and on the configuration of the twenty-seven lines of a cubic surface, which are in various ways highly characteristic. As models of analytical skill they are admirable; and as helps to the understanding of the geometrical figures with which they deal, they are of great service. But it is curious to see how chary the author is in giving illustrative diagrams. There are, indeed, two figures in the paper on the surface of centres; but why, we ask, did Cayley not give a series of contour lines of the surface? or again, with still more reason, in the case of Steiner's surface? Then the paper on the twenty-seven lines of a cubic surface is so quaint in its topsy-turveydom as almost to suggest Mr. W. S. Gilbert as joint author. Here we have a projective configuration which may be realised with the help of a bundle of sticks and without any measurement whatever. What Cayley did was to take a model by Dr. Wiener, measure approximately the coordinates of a number of points upon it, thence find the approximate equations of the lines, and finally adjust the equations so as to satisfy the geometrical conditions! Of course there is reason in this seeming perversity: by the projective method it is not easy to get a convenient arrangement of the sticks, whereas Cayley's equations make it possible to construct a string model on a cardboard frame without a tiresome series of preliminary experiments.

The poristic polygons of Poncelet appear to have had for Cayley a perennial charm: we have here two papers suggested by Poncelet's results; one "On the porism of the in-and-circumscribed polygon, &c.," which treats of the original problem, and the other "On the problem of the in-and-circumscribed triangle," which really deals with a rather different and more general theory. Cayley, like many others, does not seem to have been aware (at least in 1871) that the complete algebraical solution of the Poncelet problem was published in 1863 in a paper by M. Moutard, which formed part of the appendix to Poncelet's "Applications d'Analyse à la Géométrie." Not only is this so, but, as Halphen pointed out, this paper contains the first fully satisfactory treatment of the multiplication of the argument in elliptic functions.

Before passing on from the geometrical papers, attention should be called to the very interesting series of notes on the mechanical description of curves. This is a promising field of research, and the results could hardly fail to be of interest, especially to those who like to see the deductions of theory embodied in an actual geometrical figure. There is an æsthetic satisfaction in this contemplation: and, moreover, a really correct figure often suggests geometrical truths that would otherwise be overlooked.

Of the analytical papers the one which has been most appreciated in this country is, beyond question, the short paper "On the theory of the singular solutions of differential equations of the first order" (*Messenger*, vol. ii. (1873) pp. 6–12). Here Cayley's power of giving to analysis a geometrical interpretation appears to the best advantage. If we have an algebraical relation $f(x, y, p) = 0$ in which p enters to the degree s, then this associates with any point (x, y) a series of s (real or imaginary) directions corresponding to the different values of p: in other words, the differential equation really expresses that the plane of reference is covered with ∞^2 tiny s-rayed stars. The primitive $\phi(x, y, c) = 0$ gives a family of ∞^1 curves each made up of ∞^1 selected rays. Now if we eliminate p from $f(x, y, p) = 0$, $\partial f / \partial p = 0$, we obtain a locus of points (x, y) at each of which two rays coincide in direction; where this happens either two consecutive curves $\phi(x, y, c) = 0$ touch, or

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two non-consecutive curves touch, or (x, y) is a cusp or point of self-contact of one particular curve $\phi(x, y, c)=0$. Thus we may have the envelope of the family of curves, a tac-locus, or a locus of cusps or of points of self-contact. On the other hand if we eliminate c from $\phi(x, y, c) = 0$ and $\partial\phi/\partial c = 0$, we get the locus of intersection of consecutive curves ϕ: this may include besides the envelope proper, a locus of nodes, of cusps, or of multiple points of higher order (as, for instance, points of self-contact or triple points). The only outstanding difficulty is the degree of multiplicity in which the singular loci, distinct from the envelope, are involved in the two discriminants.

There are six papers on the transformation of elliptic functions, the most important being No. 578. This contains an exposition of the Jacobian theory, Sohnke's modular equations with additions, and a discussion of the singularities of some of the modular curves. It is remarkable that Cayley, like Kronecker, adhered firmly to Jacobian methods, and never seems to have worked with the Weierstrassian forms. Perhaps just now there is a rather exaggerated tendency in the other direction: as Prof. Klein has pointed out, both theories are self-consistent and form, in a sense, the first and second stages in a complete discussion of periodic functions.

There is not very much about invariants and co-variants; No. 525 is an interesting example of a quadratic transformation, and the papers on "trees," although ostensibly intended for application to chemistry, were suggested by the invariant calculus.

In arithmetic there is a table of reduced binary cubics with their Hessians, which is a development of Arndt's results. Cayley gives the composition tables for the Hessians.

Volume ix. contains eleven papers dealing more or less with astronomy and dynamics; and it may be worth while to notice that this volume also contains a reprint of the British Association "Report on Mathematical Tables."

Many interesting special points suggest themselves to the reader: thus, to mention only three, very different in character, the very simple and pretty proof of Vandermonde's theorem (viii. p. 465) might very well find a place in an elementary text-book of algebra; we are told (*ibid.*, p. 188) how a theoretical error was detected by a numerical calculation; and (*ibid.*, p. 397) there is an unverified conjecture that every surface of negative deficiency may be derived by a rational transformation from a cone whose deficiency is equal to that of the surface with its sign changed. G. B. M.

EXPERIMENTAL PHYSICS.

The Outlines of Physics. By Prof. E. L. Nichols. Pp. xi + 452. (London: Macmillan and Co., Ltd., 1897.)

Lessons in Elementary Practical Physics. Vol. iii. Part i. *Practical Acoustics.* By C. L. Barnes. Pp. x + 214. (London: Macmillan and Co., Ltd., 1897.)

THE first of these books, as the author explains in his preface, is an attempt to "outline a short course in physics which shall be a fair equivalent for the year of advanced mathematics now required for entrance to many colleges"; and he proceeds to point out that if physics is to possess much disciplinary value, it must be taught by laboratory methods. Experimental work thus finds a prominent place in his book, which may, in fact, be roughly described as a series of experiments, mostly suitable for repetition by young students, connected by short discussions of a theoretical character.

With the author's object we imagine that most teachers of physics will cordially sympathise. That experiment is the means whereby a knowledge of physics should be acquired by beginners, is as clear now-a-days as it is that the means itself is open to improvement—at any rate, in its early stages. Whether the author has made the most of his opportunity is, however, less certain. Much of his work is excellent: the experiments are, for the most part, well chosen and clearly described; but after a careful perusal of his book, one's prevailing impression is that he has attempted to include too much.

A book of this kind is, of course, largely taken up with description of experimental procedure; but the space is often further occupied with matter which might, in our opinion, be left until a later stage in the student's career. Such questions as X-rays, tests for and theory of colour-blindness, interference and polarisation of light, are too large for more than the briefest notice, and might therefore just as well have been omitted altogether; especially when, to mention one instance out of many, curved mirrors are dismissed with a far too scanty discussion, and no special experimental illustrations at all. It would, in our opinion, have been better to develop further the experimental treatment of the simpler parts of physics at the expense of these more elaborate phenomena. It is only in places, however, that the work is affected by this fault; and the same may be said of an occasional laxness of expression which will probably lead to mistakes on the part of young readers where it occurs. Taken as a whole, the book forms a useful addition to the elementary text-books on practical physics.

We have noticed a few points that rather need alteration. In the figure of the apparatus for determining the heat of vapourisation of water (p. 172), the long tube connecting flask and calorimeter should be provided with a trap for the steam condensed in it. The statement in italics on p. 213, that "various bodies can be brought by friction (*i.e.* by doing work upon them) into a condition such that they attract and are attracted," is rather misleading. It is, of course, the work done in pulling the rubber and rubbed object apart which should be emphasised. On p. 337, in the figure illustrating the motions of the air in sound waves, the arrows want altering; on pp. 308 and 310, misprints of iron for ion, and ammonium for ammonia, respectively, occur; and on p. 99, in the last column the decimal point has gone astray.

The general get-up of the book is, as one would expect, excellent; and the diagrams, which are mostly by Mrs. Nichols, are very clear and well executed. We may add that the work is almost wholly non-mathematical.

The second of the two books named at the head of this notice, forms the first part of vol. iii. of the "Elementary Practical Physics" series begun in 1885 by Prof. Balfour Stewart and Mr. W. W. Haldane Gee.

With the rapid development of the teaching of physics by laboratory methods, now in progress, has arisen the

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